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LETTER TO THE EDITOR

The correspondence of two definitions of coherent states in a particular system

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Abstract. Many generalizations of the harmonic oscillator coherent states are known in the literature. One such generalization by Nieto has been applied to a particular system known as the 'harmonic oscillator with centripetal barrier'. In this paper, it is shown that Nieto's generalization in this system may be described in terms of a recent generalization due to Klauder. This is of considerable interest since the approaches to generalizing coherent states differ significantly.

1. Introduction

Since the first introduction of coherent states into the literature, the idea to generalize harmonic oscillator coherent states for other systems has been present [1]. Many generalizations have been proposed each exploiting one or more of the appealing properties of the harmonic oscillator coherent states. Although Nieto's and Klauder's generalizations draw upon different properties of the harmonic oscillator coherent states, one may be described in terms of the other in the specific case described below.

Nieto's generalization of the harmonic oscillator coherent states [2] focuses principally on the fact that harmonic oscillator coherent states are minimum uncertainty states, satisfying equality in Heisenberg's uncertainly relation for the standard $\hat{x}$ and $\hat{p}$ operators. In this generalization, the coherent states satisfy minimum uncertainty for a new pair of operators $\hat{X}$ and $\hat{P}$ through which the potential 'appears' like the harmonic oscillator. The thrust in defining states according to this scheme is to provide states which remain coalesced for the longest possible time, and hence follow corresponding classical trajectories for the longest possible time via Ehrenfest's relations.

Klauder's generalization [3] focuses on the form of the eigenstate expansion which guarantees that the harmonic oscillator coherent states remain temporally stable: that the time evolution may be described by evolving the parameters to the coherent state itself. This construction prescribes the form of the eigenstate expansion of the coherent state in the potential so that the states are temporally stable by their very definition.

Klauder's generalization also provides a resolution of the identity for the set of coherent states. Note that Nieto's generalization does not provide a resolution of the identity in general, although there are several strong plausibility arguments to suggest that one should exist: if a resolution of the identity exists, it must be found case by case.

The system studied in this paper is called the harmonic oscillator with centripetal barrier due to a relationship to the radial part of the three-dimensional isotropic harmonic oscillator with non-zero angular momentum. Nieto's generalized minimum uncertainty coherent states in this system have long been known [2], and they have been shown to satisfy a resolution of the identity†. This property permits a relationship to be drawn between the coherent states of Nieto and Klauder in this case, as outlined below.

2. Generalized minimum uncertainty coherent states in the harmonic oscillator with centripetal barrier

The potential for the system in question is

$$V(x) = U_0 \left(\frac{1}{z} - z \right)^2 \tag{1}$$

in which the length scale is via $z = ax$ for some positive a, and the energy scale is given by

$$U_0 = \frac{\hbar^2 a^2}{2m} \lambda(\lambda + 1) \tag{2}$$

for $\lambda > 0$. The Hamiltonian eigenstates are given by

$$|n\rangle = \left(\frac{2a\nu\Gamma(n+1)}{\Gamma(n+\lambda+\frac{3}{2})} \right)^{1/2} \mathrm{e}^{-y/2} y^{\lambda+1} L_n^{(\lambda+1/2)}(y) \qquad n = 0, 1, 2 \ldots \tag{3}$$

where $\nu = \sqrt{\lambda(\lambda+1)}$ and $y = \nu z^2$, each eigenstate with energy

$$E_n = \frac{\hbar^2 a^2}{2m} (\nu(4n + 2\lambda + 3) - 2\nu^2). \tag{4}$$

Following Nieto [2], new position and momentum operators are given by

$$\hat{X} = \hat{z}^2 - \left(1 + \frac{H}{2U_0} \right) \qquad \hat{P} = a^2(\hat{x}\hat{p} + \hat{p}\hat{x}). \tag{5}$$

The generalized minimum uncertainty states, parametrized by the complex valued β,

$$\beta = \frac{\nu}{2}\langle X \rangle + \frac{\mathrm{i}}{4}\langle P \rangle \tag{6}$$

are given by

$$|\beta\rangle = \left(\frac{\mathrm{e}^{\mathrm{i}\mathrm{Re}\,(\beta)} \beta^{\lambda+1/2}}{I_{\lambda+1/2}(2|\beta|)} \right)^{1/2} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{\sqrt{\Gamma(n+1)\Gamma(n+\lambda+\frac{3}{2})}} |n\rangle \tag{7}$$

in which I_ν is the modified Bessel function. These states satisfy a resolution of the identity given by

$$\hat{\mathbf{1}} = \int \mathrm{d}^2\beta \; f(|\beta|)|\beta\rangle\langle\beta| \tag{8}$$

in which $\mathrm{d}^2\beta = \mathrm{dRe}\,(\beta)\,\mathrm{dIm}\,(\beta)$ and the integration is over the entire complex plane. The function $f(\rho)$ is given by

$$f(\rho) = \frac{K_{\lambda+1/2}(2\rho) I_{\lambda+1/2}(2\rho)}{8\pi} \tag{9}$$

where K_μ is the other modified Bessel function.

† A detailed background may be found in the references in [2].

3. Klauder's construction

In the briefest possible terms, Klauder's construction depends on a function $\rho(u)$. In a non-degenerate system with eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$, and eigenenergies E_n these states are given by

$$|s, \gamma\rangle = M(s^2) \sum_{n=0}^{\infty} \frac{s^n \mathrm{e}^{-\mathrm{i}\gamma E_n/\hbar}}{\sqrt{\rho_n}} |n\rangle \tag{10}$$

depending on the two real parameters, $s \geqslant 0$, and γ. In this expression, $M(u)$ is a normalizing function, and ρ_n is the nth moment of ρ,

$$\rho_n = \int_0^{\infty} u^n \rho(u)\,\mathrm{d}u. \tag{11}$$

These states satisfy the resolution of the identity

$$\hat{\mathbf{1}} = \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_0^{\infty} \mathrm{d}s^2\, k(s^2) \int_{-\Gamma}^{\Gamma} \mathrm{d}\gamma\, |s, \gamma\rangle\langle s, \gamma| \tag{12}$$

where the measure function $k(u)$is given by

$$\rho(u) = M^2(u)k(u). \tag{13}$$

4. One in terms of the other

As an initial argument, note that in their respective eigenstate expansions equations (7) and (10), the change in phase of the coefficient between successive terms is constant, in one case due to successive powers of a complex number, and in the other since the eigenenergies, equation (4), are equally spaced.

If Nieto's states given above may be described in terms of Klauder's construction, then the normalizing function $M(s^2)$ must be at least proportional to the leading part of equation (7). Also, Klauder's $k(s^2)$ must be at least proportional to the function $f(|\beta|)$ given by Nieto, and if so, $\rho(u)$ is provided by equation (13). A consistency check will then be provided by calculating the moments ρ_n which should correspond to the denominator within the sum of equation (7), i.e.,

$$\rho_n \propto \Gamma(n+1)\Gamma(n+\lambda+\tfrac{3}{2}). \tag{14}$$

To begin, the integration over the complex β plane may be expressed as

$$\int \mathrm{d}^2\beta = \int_0^{\infty} |\beta|\,\mathrm{d}|\beta| \int_0^{2\pi} \mathrm{d}\theta \tag{15}$$

where θ is being used as the phase angle of β. Accordingly, the expression equation (8) may be rewritten

$$\hat{\mathbf{1}} = \int_0^{\infty} \mathrm{d}|\beta| \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{|\beta| K_{\lambda+1/2}(2|\beta|) I_{\lambda+1/2}(2|\beta|)}{4} |\beta\rangle\langle\beta|. \tag{16}$$

Since the dependence of β on θ is 2π-periodic, integration as written with respect to θ corresponds to integration with respect to γ, so that the two may be identified.

Relating normalization functions,

$$M^2(s^2) = \frac{|\beta|^{\lambda+1/2}}{I_{\lambda+1/2}(2|\beta|)} \tag{17}$$

from equation (7). Any further progress requires the assumption $s^2 = |\beta|^p$ for some power p so that

$$M^2(u) = \frac{u^{(\lambda+1/2)/p}}{I_{\lambda+1/2}(2u^{1/p})}. \tag{18}$$

Now identifying $k(u)$ with the rest of the measure in equation (16) with integration over θ disregarded yields

$$k(u) = \frac{u^{2/p-1}K_{\lambda+1/2}(2u^{1/p})I_{\lambda+1/2}(2u^{1/p})}{4p}. \tag{19}$$

From equation (13), (19) leads to

$$\rho(u) = \frac{1}{4p}u^{\lambda/p+5/2p-1}K_{\lambda+1/2}(2u^{1/p}). \tag{20}$$

The moments of this function are given by

$$\rho_n = \frac{1}{4p}\int_0^\infty u^{n+\lambda/p+5/2p-1}K_{\lambda+1/2}(2u^{1/p})\,\mathrm{d}u. \tag{21}$$

This is a known integral [4] which yields

$$\rho_n = \frac{1}{16}\Gamma\left(\frac{p}{2}n + \lambda + \frac{3}{2}\right)\Gamma\left(\frac{p}{2}n + 1\right). \tag{22}$$

The reader will note that this does correspond to the appropriate part of Nieto's construction, equation (14), with $p = 2$. Thus, with

$$\rho(u) = \tfrac{1}{8}u^{\lambda/2+1/4}K_{\lambda+1/2}(2\sqrt{u}) \tag{23}$$

the two constructions correspond, $|\beta\rangle = \mathrm{e}^{\mathrm{i}\phi}|s,\gamma\rangle$, through the identifications ($\beta = |\beta|\mathrm{e}^{-\mathrm{i}\theta}$)

$$|\beta| = s \qquad \text{and} \qquad \theta = 2\gamma\hbar a^2\nu/m + \pi \tag{24}$$

up to an overall phase factor ϕ.

5. Some comments

It quickly becomes apparent to anyone who studies coherent states that there have been numerous attempts to generalize them. These attempts, such as those discussed in this paper, may vary wildly. Whenever possible, connections should be investigated so that the special features of the harmonic oscillator coherent states may be more precisely identified. A similar connection has been drawn by Cooper [5] with respect to the Morse oscillator.

The connection drawn in this paper is between Nieto's construction in the specific case of the harmonic oscillator with centripetal barrier, and Klauder's construction with $\rho(u)$ given by equation (23). Since Nieto's construction does not provide a resolution of the identity, any further connections between these constructions must be approached on a case by case basis. For example, it is unlikely that a similar connection may be made between Klauder's construction and Nieto's construction in the Pöschl–Teller potential [6, 7] since no resolution of the identity is known in this case and the phase angle of the eigenstate coefficients is somewhat obscure.

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